On Certain Properties of the Plane Cubic Curve in Relation to the Circular Points at Infinity.

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PART I.—*On some Methods of Generating the Plane Cubic Curve.*

I propose to investigate here some methods of generating a plane cubic curve. I begin by obtaining the cubic as a locus of a point P as follows: If perpendiculars be drawn from P on the sides of a given triangle, then the circle passing through the feet of these perpendiculars cuts orthogonally a fixed circle. A triangle involves six constants and a circle three, so that we have nine constants, which is the number involved in the equation of the general cubic.

I observe that if we describe the conic with P as a focus and touching the sides of the triangle, then the circle passing through the feet of the perpendiculars from P on the sides of the triangle is the circle described on the transverse axis of the conic as diameter. Using trilinear coordinates and expressing that the product of the perpendiculars from the foci on a tangent is constant, we get the tangential equation of the conic in the form

$$(\lambda\alpha + \mu\beta + \nu\gamma)(\lambda\alpha' + \mu\beta' + \nu\gamma') \\ - B^2(\lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C) = 0. \quad (1)$$

Now, if this conic touch the sides of the triangle, the coefficients of λ^2, μ^2, ν^2 must vanish. We thus have $\alpha\alpha' = \beta\beta' = \gamma\gamma' = B^2$. Hence, the tangential equation of the conic touching the lines α, β, γ and having the points α, β, γ as a focus, is

$$\alpha(\beta^2 + \gamma^2 + 2\beta\gamma\cos A)\mu\nu + \beta(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B)\nu\lambda \\ + \gamma(\alpha^2 + \beta^2 + 2\alpha\beta\cos C)\lambda\mu = 0. \quad (2)$$

I now obtain the equation of the director circle of this conic. The coordinates λ, μ, ν are proportional to $a\varpi_1$, $b\varpi_2$, $c\varpi_3$ respectively, where a, b, c are the sides of the triangle and $\varpi_1, \varpi_2, \varpi_3$ are the perpendiculars from the vertices of the

triangle on a line. Making this substitution, and putting $\varpi_1 = p - x_1 \cos\omega - y_1 \sin\omega$, $\varpi_2 =$ etc., and then putting $p = x\cos\omega + y\sin\omega$ for the tangents drawn to the conic (2) from x, y, we get an equation determining the directions of those tangents. Putting the sum of the coefficients of $\cos^2\omega$ and $\sin^2\omega$ equal to nothing, we get the condition that the tangents drawn to the conic from x, y should be at right angles to each other; that is, the condition that x, y should lie on the director circle of the conic. We have, then, for the equation of the director circle,

$$S = bc\alpha(\beta^2 + \gamma^2 + 2\beta\gamma\cos A)S_1 + ca\beta(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B)S_2 + ab\gamma(\alpha^2 + \beta^2 + 2\alpha\beta\cos C)S_3 = 0, \quad (3)$$

where S_1, S_2, S_3 are the circles described on the three sides as diameters respectively, viz.: $S_1 = (x - x_2)(x - x_3) + (y - y_2)(y - y_3)$, etc.

Taking the origin of the Cartesian coordinates at the centre of the conic, we may write

$$S = (a\alpha + b\beta + c\gamma)(a\beta\gamma + b\gamma\alpha + c\alpha\beta)(x^2 + y^2 - A^2 - B^2),$$

where A, B are the principal semiaxes of the conic; that is,

$$S = 2\Delta\Sigma(x^2 + y^2 - A^2 - B^2), \quad (4)$$

where Δ is the area of the triangle and $\Sigma = a\beta\gamma + b\gamma\alpha + c\alpha\beta$, so that $\Sigma = 0$ is the equation of the circumscribing circle. Again, let one focus satisfy the equation $a\alpha + b\beta + c\gamma = 2\Delta$, then, substituting $\frac{B^2}{\alpha}$, $\frac{B^2}{\beta}$, $\frac{B^2}{\gamma}$ for α, β, γ respectively, we have

$$B^2\Sigma = 2\Delta\alpha\beta\gamma. \quad (5)$$

Hence, from (4) we get

$$S + 4\Delta^2\alpha\beta\gamma = 2\Delta\Sigma(x^2 + y^2 - A^2). \quad (6)$$

But $x^2 + y^2 - A^2 = 0$ is the circle described on the transverse axis of the conic as diameter; that is, the circle passing through the feet of the perpendiculars from α, β, γ on the sides of the triangle. Now, if this circle cut orthogonally a fixed circle, its coefficients are connected by a linear relation. Hence, expressing that the coefficients of $S + 4\Delta^2\alpha\beta\gamma$ are connected by a linear relation, we have

$$L\alpha(\beta^2 + \gamma^2 + 2\beta\gamma\cos A) + M\beta(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B) + N\gamma(\alpha^2 + \beta^2 + 2\alpha\beta\cos C) + 2P\alpha\beta\gamma = 0, \quad (7)$$

where L, M, N, P are constants determining the position of the fixed circle. It may be observed that

$$L = bc\,(S_1 - k^2), \quad M = ca\,(S_2 - k^2), \quad N = ab\,(S_3 - k^2), \quad P = 2\Delta^2, \qquad (8)$$

where k is the radius of the fixed circle, and S_1, S_2, S_3 are the squares of the tangents drawn from the centre of the fixed circle to the circles described on the sides of the triangles as diameters respectively. Now, (7) represents a cubic passing through the vertices of the triangle, so that the points where the curve meets the sides again lie on a line, viz.:

$$MN\alpha + NL\beta + LM\gamma = \delta = 0. \qquad (9)$$

The curve (7), then, can be written in the form

$$\delta\,(MN\beta\gamma + NL\gamma\alpha + LM\alpha\beta) + \{2LMN(L\cos A + M\cos B + N\cos C + P) - M^2N^2 - N^2L^2 - L^2M^2\}\,\alpha\beta\gamma = 0, \qquad (10)$$

so that, dividing by $\alpha\beta\gamma\delta$, the curve is of the form

$$\frac{l}{\alpha} + \frac{m}{\beta} + \frac{n}{\gamma} + \frac{p}{\delta} = 0, \qquad (11)$$

where l, m, n, p are constants. Now, I observe that the Hessian of the cubic

$$A_1 x_1^3 + A_2 x_2^3 + A_3 x_3^3 + A_4 x_4^3 = 0, \qquad (12)$$

where A_1, A_2, A_3, A_4 are constants and $x_1 + x_2 + x_3 + x_4 = 0$ identically is

$$\frac{1}{A_1 x_1} + \frac{1}{A_2 x_2} + \frac{1}{A_3 x_3} + \frac{1}{A_4 x_4} = 0, \qquad (12)$$

and that corresponding points on the latter cubic are connected by the relations

$$A_1 x_1 x_1' = A_2 x_2 x_2' = A_3 x_3 x_3' = A_4 x_4 x_4'. \qquad (13)$$

We see thus that points on the cubic (7) such that $\alpha\alpha' = \beta\beta' = \gamma\gamma'$, viz., foci of the conic (2) are corresponding points on the curve, namely, points such that the tangents thereat intersect on the cubic. Now, the cubic (7) is the Hessian of a cubic of the form

$$MN\alpha^3 + NL\beta^3 + LM\gamma^3 + \theta\delta^3 = 0, \qquad (14)$$

so that the polar line, that is, the pole of one point with regard to the polar conic of the other, of the circular points, for which $\alpha\alpha' = \beta\beta' = \gamma\gamma'$ is $\delta = 0$, viz., the line passing through the three points on the curve corresponding to the vertices of the triangle. Now, there are three cubics of which a given cubic is the

Hessian, corresponding to the three systems of corresponding points, so that the cubic can be written in the form (7) in three ways. Thus the cubic can be generated in the manner described in three ways.

A general circular cubic cannot be generated in this manner. For, if the cubic (7) is circular, $P = 0$; but then the circular points are corresponding points on the curve; that is, the double focus of the curve is on itself. In such a case the circle passing through the feet of the perpendiculars has its centre on a fixed line instead of cutting a fixed circle orthogonally, as is evident from the fact that the conic (2) then touches another fixed line. The cubic is then the locus of a point P such that the feet of the perpendiculars from P on the sides of a quadrilateral lie on a circle.

If the fixed circle satisfy a certain relation, the locus breaks up into a conic passing through the vertices of the triangle and a right line. This relation is, from (10),

$$2LMNP = M^2N^2 + N^2L^2 + L^2M^2 - 2LMN(L\cos A + M\cos B + N\cos C);$$

that is, from (8),

$$\begin{aligned}4\Delta^2(S_1 - k^2)(S_2 - k^2)(S_3 - k^2) = {} & a^2(S_2 - k^2)(S_3 - k^2)(S_1 - S_2)(S_1 - S_3) \\ & + b^2(S_3 - k^2)(S_1 - k^2)(S_2 - S_3)(S_2 - S_1) \\ & + c^2(S_1 - k^2)(S_2 - k^2)(S_3 - S_1)(S_3 - S_2). \qquad (15)\end{aligned}$$

Hence, selecting any given point as centre, we have a cubic for k^2, so that three circles satisfying the condition are determined. Again, I observe that if the cubic (7) has a node, that point is situated at the centre of one of the circles touching the sides of the triangle; for instance, for the centre of the inscribed circle the cubic is

$$L\alpha(\beta - \gamma)^2 + M\beta(\gamma - \alpha)^2 + N\gamma(\alpha - \beta)^2 = 0, \qquad (16)$$

so that, from (8), the fixed circle then satisfies the condition

$$\begin{aligned}\tfrac{1}{4}(b + c - a)(c + a - b)(a + b - c) + (b + c - a)\,S_1 & \\ + (c + a - b)\,S_2 + (a + b - c)\,S_3 - k^2(a + b + c) = 0, & \qquad (17)\end{aligned}$$

that is, it must cut orthogonally the inscribed circle, as it can easily be proved that the result of putting $k = 0$ in (17) gives the equation of the inscribed circle.

I now proceed to show that the same cubic can be generated if the circle

$$x^2 + y^2 = A^2 + B^2 - mB^2 \tag{18}$$

cut orthogonally a fixed circle.

We have from (4),

$$S = 2\Delta\Sigma\,(x^2 + y^2 - A^2 - B^2),$$

where, from (3), $S = bc\alpha\,(\beta^2 + \gamma^2 + 2\beta\gamma\cos A) + ca\beta\,(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B)$
$+ ab\gamma\,(\alpha^2 + \beta^2 + 2\alpha\beta\cos C)$;

also from (5), $B^2\Sigma = 2\Delta\alpha\beta\gamma$, where $\Sigma = a\beta\gamma + b\gamma\alpha + c\alpha\beta$. Hence,

$$S + 4m\Delta^2\alpha\beta\gamma = 2\Delta\Sigma\,\{x^2 + y^2 - A^2 - B^2 + mB^2), \tag{19}$$

so that to express that (18) cuts a fixed circle orthogonally, we write down a linear relation connecting the coefficients of the circle $S + 4m\Delta^2\alpha\beta\gamma = 0$, when we obtain a cubic of the form (7), where

$$L = bc\,(S_1 - k^2), \quad M = ca\,(S_2 - k^2), \quad N = ab\,(S_3 - k^2), \quad P = 2m\Delta^2. \tag{20}$$

From these equations it is easy to see that, if m be considered indeterminate, the fixed circle is not necessarily given, but may be replaced by any other circle passing through two given points, its centre lying on the line

$$La\,(S_2 - S_3 + Mb\,(S_3 - S_1) + Nc\,(S_1 - S_2) = 0. \tag{21}$$

Hence, making the radius vanish, we have from (20),

$$\frac{S_1}{La} = \frac{S_2}{Mb} = \frac{S_3}{Nc}, \tag{22}$$

which determines two points mutually inverse with regard to the polar circle of the triangle, as the latter circle is the Jacobian circle of S_1, S_2, S_3. We see thus that two circles of the system (18) can be determined, that is, two constant values of m can be assigned, so that the circle (19) passes through a fixed point. Again, if a certain condition be satisfied by the fixed circle similar to (15), the cubic breaks up into a line and a conic. This condition, corresponding to a given line, supplies a relation connecting m with the fixed circle.

Suppose $m = 2$, then the circle (18) is $x^2 + y^2 = A^2 - B^2$, that is, it is the circle described on the line joining the foci of the conic as diameter. Expressing then that $S + 8\Delta^2\alpha\beta\gamma$ cuts a fixed circle orthogonally, we obtain a general cubic of the form (7). Now, the foci of the conic are corresponding points on the cubic, so that we see that the circle described on the line joining a pair of cor-

responding points as diameter cuts orthogonally a fixed circle. This can be proved otherwise thus: A pair of corresponding points on the curve are conjugate with regard to all the polar conics of the cubic of which the given curve is the Hessian. Now, the equation of the polar conics being of the form $lU_1 + mU_2 + nU_3 = 0$, one of the system is a circle. But when two points are conjugate with regard to a circle, U say, the circle described on the line joining them as diameter cuts U orthogonally. In the particular case, it may be observed, when the Cayleyan has a focus on the cubic, at P say, that is, when U breaks up into factors, all the pairs of corresponding points subtend right angles at P.

I now proceed to find the locus of the centre of the conic (2), when the circle described on the transverse axis as diameter cuts orthogonally a fixed circle, or, in other words, the locus of the middle point of pairs of corresponding points on a given cubic. From the first point of view, as I shall show, this may be considered as another method of generating a general cubic. Let a, b be half the principal axes of the conic, and α, β, γ the perpendiculars from the centre on the sides of the triangle. Then

$$\alpha^2 = a^2 \cos^2(\theta - \alpha) + b^2 \sin^2(\theta - \alpha), \tag{23}$$

and similar values for β, γ, where α, β, γ, θ are the angles which the perpendiculars and the axis major make with a fixed line respectively. Eliminating θ and b, we obtain

$$\sin A \sqrt{(a^2 - \alpha^2)} + \sin B \sqrt{(a^2 - \beta^2)} + \sin C \sqrt{(a^2 - \gamma^2)} = 0, \tag{24}$$

where A, B, C are the angles of the triangle. Now, if the circle described on the axis major as diameter cuts orthogonally a circle $S = x^2 + y^2 + \text{etc.}$, we have $a^2 = S$, so that (24) becomes

$$\sin A \sqrt{(S - \alpha^2)} + \sin B \sqrt{(S - \beta^2)} + \sin C \sqrt{(S - \gamma^2)} = 0, \tag{25}$$

which, at first sight, appears to represent a quartic, but, being divisible by the line at infinity, reduces to a cubic, as $S - \alpha^2$, $S - \beta^2$, $S - \gamma^2$ are easily seen to be parabolæ. The cubic (25) is, in fact, the envelope of the conic

$$l(S - \alpha^2) + m(S - \beta^2) + n(S - \gamma^2) = 0, \tag{26}$$

subject to the condition that it is a parabola, viz.:

$$mn \sin^2 A + nl \sin^2 B + lm \sin^2 C = 0. \tag{27}$$

We can obtain the equation of this cubic in another form, thus: Eliminating θ and a from (23), we get

$$\sin A\sqrt{(\alpha^2-b^2)}+\sin B\sqrt{(\beta^2-b^2)}+\sin C\sqrt{(\gamma^2-b^2)}=0, \tag{27}$$

then, observing that, since the director circle of the conic cuts orthogonally the polar circle P of the triangle, we have $a^2+b^2=P$, so that if $S-P=\delta$, namely, the radical axis of S and P, we get

$$\sin A\sqrt{(\alpha^2+\delta)}+\sin B\sqrt{(\beta^2+\delta)}+\sin C\sqrt{(\gamma^2+\delta)}=0, \tag{28}$$

which, being divisible by the line at infinity, represents a cubic. This curve, it may be observed, passes through the points where the lines joining the middle points of the sides meet the circle S and the radical axis of S and P.

Projecting, we find that the polar of a fixed line with regard to A, B is a cubic, where A, B are corresponding points on a given cubic.

In the case of the circular cubic, we may find the locus of the middle point of corresponding points thus: Let the curve be

$$lv(u^2+c^2)+mu(v^2+c^2)+n(u^2+v^2)+2puv=0, \tag{29}$$

where u, v are circular coordinates, then corresponding points are such that $uu'=vv'=c^2$. Hence, for the middle point of corresponding points we may write

$$2u=u'+\frac{c^2}{u'}, \quad 2v=v'+\frac{c^2}{v'}, \tag{30}$$

where u', v' lie on (29). Hence, we obtain

$$\left(lu+mv+\frac{n}{c^2}uv+p\right)^2=\frac{n^2}{c^4}(u^2-c^2)(v^2-c^2), \tag{31}$$

which represents a circular cubic with the points $u^2=v^2=c^2$ as foci. Thus the locus (31) is the transformation of the given curve (29) by a substitution (30) in which angles are preserved. The substitution is, in fact, a transformation from polar to elliptic coordinates, viz.:

$$r=\mu+\sqrt{(\mu^2-c^2)}, \quad c\cos\theta=\nu, \tag{32}$$

where
$$cx=\mu\nu, \quad cy=\sqrt{\{(\mu^2-c^2)(c^2-\nu^2)\}}, \tag{33}$$

and r, θ are polar coordinates. It may be observed that the anharmonic function of the locus can be expressed in terms of the similar function for the given

curve. For it is easy to see that the values of u corresponding to the foci of (29) are given by an equation of the form

$$(u^2 + c^2 - 2\alpha u)(u^2 + c^2 - 2\beta u) = 0, \tag{34}$$

so that the foci of (31) are given by

$$(u - \alpha)(u - \beta)(u^2 - c^2) = 0, \tag{35}$$

but the anharmonic functions of both these biquadratics are expressible in terms of $\dfrac{c(\alpha - \beta)}{c^2 - \alpha\beta}$.

If $n = 0$ in (29), the double focus is at the origin on the curve, and the locus (31) reduces to the right line

$$lu + mv + p = 0, \tag{36}$$

as we have seen already otherwise. That is, when the cubic is not circular, the locus reduces to a right line, if two asymptotes intersect on the curve.

I now proceed to show what geometrical relations are satisfied when a focus of the conic (2) lies on a general cubic passing through the vertices of the triangle of reference. Let Δ be the area of the triangle formed by the foci P, P' of the conic and a fixed point α', β', γ'; then Δ is proportional to

$$\begin{vmatrix} \dfrac{b^2}{\alpha}, & \dfrac{b^2}{\beta}, & \dfrac{b^2}{\gamma} \\ \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \end{vmatrix} = \frac{b^2}{\alpha\beta\gamma} V,$$

where

$$V = \alpha'\alpha(\beta^2 - \gamma^2) + \beta'\beta(\gamma^2 - \alpha^2) + \gamma'\gamma(\alpha^2 - \beta^2). \tag{37}$$

Again, let t be the length of the tangent drawn from a fixed point to the circle passing through the feet of perpendiculars from P on the sides of the triangle of reference, then, if we have

$$t^2 - k^2 = \lambda\Delta, \tag{38}$$

where k and λ are constants, we see from (7) and (37) that the locus of P is

$$L\alpha(\beta^2 + \gamma^2 + 2\beta\gamma\cos A) + M\beta(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B)$$
$$N\gamma(\alpha^2 + \beta^2 + 2\alpha\beta\cos C) + 2P\alpha\beta\gamma + \lambda' V = 0, \tag{39}$$

and this obviously represents a cubic circumscribing the triangle and satisfying no further relation with it, for we have six constants at our disposal,

namely, k, λ and the four depending upon the two fixed points. The locus of the other focus is evidently obtained by changing the sign of V in (39), as V becomes $-\dfrac{V}{\alpha^2\beta^2\gamma^2}$ when we substitute for α, β, γ their reciprocals. This mode of generation of the cubic holds also if we substitute the circle described on PP' as diameter for the circle passing through the feet of the perpendiculars from P; in fact, we might substitute any circle of the system (19).

If we want to generate a circular cubic we should take $P = 0$ in (39) and make V pass through the circular points. The first condition makes the circle the director circle of the conic (2), and the second requires that the point α', β', γ' should be at infinity. Thus the relation (38) becomes $t^2 - k^2 = \lambda\delta$, (40), where t is the length of the tangent drawn from a fixed point to the director circle of the conic and δ is the projection of PP' on a fixed line. It may be observed that k may be made to vanish in (40) without loss of generality. A general circular cubic is also generated when the circle described on PP' as diameter cuts a given line at an angle whose cosine is proportional to the cosine of the angle between PP' and a fixed direction. Similarly, a general cubic is generated when the circle on PP' as diameter meets a given circle at an angle ϕ, so that $\cos\phi$ is proportional to the perpendicular from a fixed point on PP'.

It may be worth while considering the case in which the two fixed points involved in (38) coincide for the circle described on PP' as diameter. It is easy to see then that the cubic is the locus of the focus P of a conic touching the sides of a triangle, subject to the condition that the circle described through the foci P, P' so as to contain a given angle, cuts orthogonally a fixed circle. The equation of the locus is found to be

$$\begin{aligned}\frac{\alpha'\alpha}{\sin A}\{(\beta^2+\gamma^2)\cos A + 2\beta\gamma\} &+ \frac{\beta'\beta}{\sin B}\{(\gamma^2+\alpha^2)\cos B + 2\gamma\alpha\}\\ &+ \frac{\gamma'\gamma}{\sin C}\{(\alpha^2+\beta^2)\cos C + 2\alpha\beta\}\\ &- \frac{(S'-k^2)}{2R\sin A\sin B\sin C}(\alpha\sin A + B\sin B + \gamma\sin C)\\ &\qquad\qquad\times(\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C)\\ &+ m\{\alpha'\alpha(\beta^2-\gamma^2) + \beta'\beta(\gamma^2-\alpha^2)\gamma'\gamma(\alpha^2-\beta^2)\} = 0, \quad (41)\end{aligned}$$

where α', β', γ' are the coordinates of the centre of the fixed circle, S' is the square of the tangent drawn from its centre to the circumscribing circle, and k is

its radius, while m is the cotangent of the given angle. This form (41) contains ten constants, so that a cubic can be generated in this manner in a singly infinite number of ways. If $k=0$, the foci P, P' subtend a constant angle at a fixed point, and the equation (41) then involving nine constants, the cubic can be generated in this manner in a finite number of ways.

I observe that a circular cubic can be written in the form (29) in three ways, the origin in each case being one of the points corresponding to the real point at infinity. In connection with this form, I proceed to investigate a mode of generation of the curve. Taking rectangular Cartesian coordinates and writing a conic referred to its principal axes in the form

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0. \tag{42}$$

the equation of the circle passing through x', y' and the points of contact of the tangents drawn from x', y' to $S=0$ is

$$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)(x^2+y^2) - \frac{xx'}{a^2}(x'^2+y'^2+c^2) - \frac{yy'}{b^2}(x'^2+y'^2-c^2) + c^2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right) = 0, \tag{43}$$

where $c^2 = a^2 - b^2$.

Hence, if this circle cut orthogonally the fixed circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0, \tag{44}$$

the locus of x', y' is

$$U = \frac{\alpha x}{a^2}(x^2+y^2+c^2) + \frac{\beta y}{b^2}(x^2+y^2-c^2) - (k^2+c^2)\frac{x^2}{a^2} - (k^2-c^2)\frac{y^2}{b^2} = 0, \tag{45}$$

which, it is easy to see, is of the same form as (29). There is no loss of generality in making the radius of (44) vanish, in which case the circle (43) passes through a fixed point. In that case, the cubic is the locus of the six vertices of a quadrilateral circumscribed about the conic, the lines drawn from the fixed point (44) to the four points of contact all making the same angle with the conic.

If the polar circle of the triangle formed by the tangents from P and their chord of contact cuts orthogonally a fixed circle, the locus of P is a cubic with

two rectangular asymptotes. For the equation of the polar circle is

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)(x^2+y^2)-\frac{2x'}{a^2}\left(a^2+b^2+\frac{c^2y'^2}{b^2}\right)x$$
$$-\frac{2y'}{b^2}\left(a^2+b^2-\frac{c^2x'^2}{a^2}\right)y+a^2+b^2+\frac{b^2}{a^2}x'^2+\frac{a^2}{b^2}y'^2=0, \qquad (46)$$

and if this cuts the circle (44) orthogonally, the locus of x', y' is

$$V=\frac{2c^2}{a^2b^2}xy(\beta x-\alpha y)+(b^2+k^2)\frac{x^2}{a^2}+(a^2+k^2)\frac{y^2}{b^2}$$
$$-2(a^2+b^2)\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}\right)+a^2+b^2=0. \qquad (47)$$

We can now generate the general cubic by means of these results, (45) and (47). Let S denote the circle (43) circumscribing the triangle formed by the tangents from P and their chord of contact, and let P denote the polar circle (46) of the same triangle, then let $\Sigma = mS + nP = 0$; that is, let Σ be a circle coaxial with S and P and with its centre dividing in a constant ratio, the line joining the centres of S and P. Now let Σ cut the given circle (44) orthogonally, then the locus of $P(x', y')$ is the general cubic $mU + nV = 0$.

I now proceed to consider another method of generating a cubic curve. Let us consider a circle passing through points on the sides of a triangle so that the lines joining them to the opposite vertices form two sets of three concurrent lines, which, it is to be observed, is only equivalent to a single condition. Using areal coordinates, the equation

$$(x+y+z)(ll'x+mm'y+nn'z)=k(a^2yz+b^2zx+c^2xy) \qquad (48)$$

represents a circle. Making this identical with

$$ll'x^2+mm'y^2+nn'z^2-(mn'+m'n)yz-(nl'+n'l)zx-(lm'+l'm)xy=0, \qquad (49)$$

where the lines joining the vertices to the intersections with the opposite sides are $lx-my=0$, $l'x-m'y$, etc., intersecting in the points mn, nl, lm; $m'n'$, $n'l'$, $l'm'$ respectively, say these are P, P', so that l, m, n; l', m', n' are the coordinates of points inverse to P, P' with regard to the triangle of reference, we obtain

$$(m+n)(m'+n')=ka^2, \quad (n+l)(n'+l')=kb^2, \quad (l+m)(l'+m')=kc^2, \qquad (50)$$

so that (48) becomes

$$(x+y+z)\{a^2(l+m)(l+n)A+b^2(m+l)(m+n)B+c^2(n+l)(n+m)C\} \\ =2(m+n)(n+l)(l+m)(a^2yz+b^2zx+c^2xy), \quad (51)$$

where $\quad A=my+nz-lx, \quad B=lx+nz-my, \quad C=lx+my-nz.$

Now, expressing that this circle (51) cuts orthogonally a given circle, we get a linear solution connecting the coefficients. We thus have a relation of the third degree connecting l, m, n, so that the inverse points of P, P' with regard to the triangle lie on a cubic circumscribing the triangle $l+m=0$, $l+n=0$, $m+n=0$, so that the points where it meets the sides again lie on a line. And it may be observed that the two points are corresponding points of the cubic. Now the triangle of reference involves six constants and the given circle involves three, so that we have the nine constants involved in a general cubic.

I now consider Grassman's method of generating a cubic. Let us consider two triangles whose vertices are A, B, C, A', B', C' respectively; then, if the lines joining P to A, B, C meet $B'C'$, $C'A'$, $A'B'$ in three collinear points, then the locus of P is the cubic

$$2\Delta\alpha'\beta'\gamma'-cp_3\alpha'\beta'\gamma-bp_2\alpha'\gamma'\beta-ap_1\beta'\gamma'\alpha=0, \quad (52)$$

where 2Δ is the area of ABC and p_1, p_2, p_3 are the perpendiculars from A, B, C respectively on $B'C'$, $C'A'$, $A'B'$. This cubic circumscribes the two triangles and furthermore passes through the intersection of corresponding sides, viz., the points AB, $A'B'$, etc. From the latter fact it can be readily deduced by means of the arguments, viz., the elliptic integrals which correspond to points on the curve, that A', B', C' must be corresponding points to A, B, C of the same system. Now, we know otherwise that the lines joining a point P of the curve to three pairs of corresponding points are in involution, so that in the same case the cubic can be written

$$AC'.BA'.CB'=AB'.BC'.CA', \quad (53)$$

where AC', etc., mean the areas of the triangles PAC', etc. Hence, when the curve is circular, we can find two relations connecting the two triangles. For, if we have the cubic $\alpha\beta\gamma=k\delta\varepsilon\zeta$, then substituting the coordinates of the circular points, we get $k=1$, $a+\beta+\gamma=\delta+\varepsilon+\zeta$, where a is the angle which α makes with a fixed line, etc., so that (53) gives $AC'.BA'.CB'=AB'.BC'.CA'$, where AC' is

the length of the line joining A, C', etc. Also, the sum of the angles between AC' and CA', BA' and AB', and CB' and BC' vanishes. Again, if the lines joining P to A, B, C respectively meet $B'C'$, $C'A'$, $A'B'$ in three points which form a triangle of given area M, the locus of P is the cubic

$$U = 2M(\alpha' - p_1)(\beta' - p_2)(\gamma' - p_3), \tag{54}$$

where U is the cubic (52). This represents a cubic passing through the points A, B, C.

Again, I consider a system of quadrilaterals with a given triangle of centres, and I seek the locus of the vertices when the circle passing through three of them cuts orthogonally a given circle. Taking the triangle of centres as the triangle of reference, and using areal coordinates, the equation of a circle is

$$(x + y + z)(lx + my + nz) = x^2 \cot A + y^2 \cot B + z^2 \cot C,$$

and expressing that this circle passes through the points $-x', y', z'$; $x', -y', z'$; $x', y', -z'$, its equation becomes

$$\begin{aligned}(x'^2 \cot A + y' \cot B + z'^2 \cot C)(x + y + z)\{(y' + z' - x')\,x \\ + (z' + x' - y')\,y + (x' + y' - z')\,z\} \\ = (y' + z' - x')(z' + x' - y')(x' + y' - z')(x^2 \cot A + y^2 \cot B + z^2 \cot C).\end{aligned} \tag{55}$$

Hence, we see that if this circle cuts orthogonally a fixed circle, the locus of the fourth vertex (x', y', z') of the quadrilateral is a general cubic curve passing through the six imaginary points where the lines joining the middle points of the sides of the triangle of centres meet the polar circle. Also, the locus of one of the three vertices is a circular cubic passing through the four points where two of the lines joining the middle points of the sides meet the polar circle. We thus have as loci of the vertices of the quadrilateral one general cubic and three circular cubics.

I consider here now a locus connected with the cubic. Let AB be a chord of the cubic perpendicular to an asymptote, and let α, β, γ be the coordinates of a point P with regard to the triangle A, B, C, where C is the point at infinity on the same asymptote, then the equation of the cubic can be written

$$(l\alpha + m\beta)\,\gamma^2 + \alpha\beta\,(l'\alpha + m'\beta) + \gamma\,(a\alpha^2 + b\beta^2 + h\alpha\beta) = 0. \tag{56}$$

Now, if α', β', γ' are the coordinates of the intersection of the perpendiculars of the triangle PAB, it can easily be shown that $\alpha\beta' = \beta\alpha' = \gamma\gamma'$. Hence, the

locus of the intersection of the perpendiculars of the triangle ABP is the cubic

$$\alpha\beta\,(l\alpha + m\beta) + \gamma^2\,(l'\alpha + m'\beta) + \gamma\,(a\alpha^2 + b\beta^2 + h\alpha\beta) = 0, \tag{57}$$

intersecting the given cubic at the point at infinity C and at six points on the circle $\gamma^2 - \alpha\beta = 0$, namely, the circle described on AB as diameter. Let $l = l'$, $m = m'$, then the cubic has, it is easy to see, two mutually perpendicular asymptotes, and the intersection of perpendiculars of the triangle PAB lies on the curve.

Further, I note the following generation of a cubic. Given the base AB of a triangle, if a point P dividing in a given ratio the line joining the centre of the circumscribed circle and the intersection of the perpendiculars lies on a hyperbola with an asymptote perpendicular to AB, then the locus of the vertex C is a cubic such that AB is a chord of the cubic perpendicular to an asymptote at the finite point where that asymptote meets the curve.

Taking the axis as the base, and perpendicular to it at the middle point, the coordinates of the point P are given by

$$x = \frac{mx'}{m+n}, \quad y = \frac{m\,(x'^2 + y'^2 - c^2) + 2n\,(c^2 - x'^2)}{2y'\,(m+n)}, \tag{58}$$

where x', y' are the coordinates of the vertex and m/n is the given ratio. Hence, if P lies on a hyperbola with an asymptote perpendicular to AB, the locus of P is the cubic

$$\{my^2 + (m - 2n)(x^2 - c^2)\}(Ax + B) + y\,(Cx^2 + Dx + E), \tag{59}$$

which is such that A, B are two points on the curve on a perpendicular to the asymptote $Ax + B = 0$, where it meets the curve.

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